

PITHAPUR RAJAHS GOVERNMENT COLLEGE(A): KAKINADA

DEPARTMENT OF MATHEMATICS

MATHEMATICAL SPECIAL FUNCTIONS

BETA AND GAMMA FUNCTIONS

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Gamma and Beta Functions

Introduction

As introduced by the Swiss mathematician Leonhard Euler in 18th century, gamma function is the extension of factorial function to real numbers. Beta function (also known as Euler's integral of the first kind) is closely connected to gamma function; which itself is a generalization of the factorial function. Both Beta and Gamma functions are very important in calculus as complex integrals can be moderated into simpler form using and Beta and Gamma function.

I Gamma Function

We define Gamma function as: $\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$

Important results

1. i. $\Gamma 1 = 1$

Proof: $\Gamma 1 = \int_0^{\infty} e^{-x} x^0 dx = -[e^{-x}]_0^{\infty} = 1$

ii. $\Gamma \frac{1}{2} = \sqrt{\pi}$

Proof: $\Gamma \frac{1}{2} = \int_0^{\infty} e^{-x} x^{-\frac{1}{2}} dx = \int_0^{\infty} e^{-t^2} t^{-1} 2t dt$, by putting $x = t^2$

$$= 2 \int_0^{\infty} e^{-t^2} dt = \sqrt{\pi}, \quad \because \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\therefore \Gamma \frac{1}{2} = \Gamma(0.5) = \sqrt{\pi} = 1.772$$

2. Reduction formula for Γn : $\Gamma(n+1) = n\Gamma n$

We have $\Gamma(n+1) = \int_0^{\infty} e^{-x} x^n dx$

$$= -[x^n e^{-x}]_0^\infty + n \int_0^\infty x^{n-1} e^{-x} dx = 0 + n\Gamma n$$

$$\therefore \Gamma(n+1) = n\Gamma n$$

$$3. \int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma n}{k^n}$$

Proof: We have $\Gamma n = \int_0^\infty e^{-t} t^{n-1} dt$

$$\text{Putting } t = kx \Rightarrow dt = kdx$$

$$\therefore \Gamma n = \int_0^\infty e^{-kx} (kx)^{n-1} kdx = k^n \int_0^\infty e^{-kx} x^{n-1} dx$$

$$\Rightarrow \int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma n}{k^n}$$

Extension of Gamma function from factorial notation

Case *i*. When n is a positive integer

We have $\Gamma(n+1) = n\Gamma n$

$$= n(n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2)$$

$$\vdots$$

$$= n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 \Gamma 1 = n!$$

$$\therefore \Gamma 2 = 1!, \Gamma 3 = 2!, \Gamma 4 = 3! \text{ etc.}$$

case *ii*. When n is a positive rational number

$\Gamma n = (n-1)(n-2) \cdots$ upto a positive number in Γ function

Illustration: $\Gamma \frac{7}{2} = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma \frac{1}{2} = \frac{15\sqrt{\pi}}{8}$

Also $\Gamma \frac{11}{4} = \frac{7}{4} \cdot \frac{3}{4} \Gamma \frac{3}{4}$

Now value of $\Gamma \frac{3}{4}$ can be obtained from table of gamma function.

case *iii*. When n is a negative rational number

Using $\Gamma(n+1) = n\Gamma n$

$$\begin{aligned} \Rightarrow \Gamma n &= \frac{\Gamma(n+1)}{n} = \frac{(n+1)\Gamma(n+1)}{n(n+1)} \\ &= \frac{\Gamma(n+2)}{n(n+1)} \\ &= \frac{\Gamma(n+3)}{n(n+1)(n+2)} \\ &\vdots \end{aligned}$$

Continuing in this manner, we get $\Gamma n = \frac{\Gamma(n+k+1)}{n(n+1)\dots(n+k)}$, where k is the least positive integer such that $(n+k+1) > 0$

Illustration: $\Gamma(-3.4) = \frac{\Gamma(-3.4+k+1)}{(-3.4)(-2.4)\dots(-3.4+k)}$, $(-3.4+k+1) > 0$

$$\Rightarrow k > 2.4 \Rightarrow k = 3$$

$$\therefore \Gamma(-3.4) = \frac{\Gamma(-3.4+4)}{(-3.4)(-2.4)(-1.4)(-0.4)} = \frac{\Gamma 0.6}{(-3.4)(-2.4)(-1.4)(-0.4)}$$

$\Gamma 0.6$ can be found using tables.

Also, to evaluate $\Gamma(-2.5)$,

$$\Gamma(-2.5) = \frac{\Gamma(-2.5+k+1)}{(-2.5)(-1.5)\dots(-2.5+k)}, \quad (-2.5 + k + 1) > 0$$

$$\Rightarrow k > 1.5 \Rightarrow k = 2$$

$$\therefore \Gamma(-2.5) = \frac{\Gamma(-2.5+3)}{(-2.5)(-1.5)(-0.5)} = \frac{\Gamma 0.5}{(-2.5)(-1.5)(-0.5)} = -\frac{1.772}{1.875} = -0.945$$

case *iv*. Γn is not defined when $n = 0$ or a negative integer

$$\text{We know } \Gamma n = \frac{\Gamma(n+k+1)}{n(n+1)\dots(n+k)}, \quad n = 0, -1, -2, \dots$$

For all $n = 0, -1, -2, \dots$, we will have a zero in the denominator

$$\text{For instance, } \Gamma 0 = \frac{\Gamma(0+k+1)}{0(1)\dots(0+k)}, \quad \Gamma(-1) = \frac{\Gamma(-1+k+1)}{(-1)(0)\dots(-1+k)}, \dots$$

Hence, we can conclude that gamma function cannot be defined for zero or negative integers.

Example 1 If n is a positive integer, show that

$$2^n \Gamma\left(n + \frac{1}{2}\right) = 1.3.5 \dots (2n-1)\sqrt{\pi}$$

$$\text{Solution: } \Gamma\left(n + \frac{1}{2}\right) = \Gamma\left(n - \frac{1}{2} + 1\right)$$

$$= \left(n - \frac{1}{2}\right) \Gamma\left(n - \frac{1}{2}\right) \quad \because \Gamma(n+1) = n\Gamma n$$

$$= \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) \Gamma\left(n - \frac{3}{2}\right)$$

$$\vdots$$

$$= \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) \left(n - \frac{5}{2}\right) \dots \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma \frac{1}{2}$$

$$= \left(\frac{2n-1}{2}\right) \left(\frac{2n-3}{2}\right) \left(\frac{2n-5}{2}\right) \dots \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}$$

$$\Rightarrow 2^n \Gamma\left(n + \frac{1}{2}\right) = 1.3.5 \dots (2n-1) \sqrt{\pi}$$

Example 2 Evaluate the following integrals

i. $\int_0^\infty e^{-x^2} x^{2n-1} dx, n > 1$

ii. $\int_0^\infty e^{-\sqrt{x}} x^{\frac{1}{4}} dx$

iii. $\int_0^\infty \frac{x^a}{a^x} dx$

iv. $\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx, n > 0$

Solution: i. We have $\Gamma n = \int_0^\infty e^{-t} t^{n-1} dt, \dots \textcircled{1}$

Putting $t = x^2$ in $\textcircled{1}$, we get

$$\Gamma n = \int_0^\infty e^{-x^2} x^{2n-2} \cdot 2x dx$$

$$\Rightarrow \int_0^\infty e^{-x^2} x^{2n-1} dx = \frac{\Gamma n}{2}$$

ii. Putting $t = \sqrt{x}$ in $\textcircled{1}$, we get

$$\Gamma n = \int_0^\infty e^{-\sqrt{x}} x^{\frac{n}{2}-\frac{1}{2}} \cdot \frac{1}{2} x^{-\frac{1}{2}} dx = \frac{1}{2} \int_0^\infty e^{-\sqrt{x}} x^{\frac{n}{2}-1} dx$$

Substituting $\frac{n}{2} - 1 = \frac{1}{4}$, i.e. $n = \frac{5}{2}$, we get

$$\Gamma\left(\frac{5}{2}\right) = \frac{1}{2} \int_0^{\infty} e^{-\sqrt{x}} x^{\frac{1}{4}} dx$$

$$\therefore \int_0^{\infty} e^{-\sqrt{x}} x^{\frac{1}{4}} dx = 2\Gamma\left(\frac{5}{2}\right) = 2 \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{2}$$

iii. Putting $a^x = e^t$ or $x \log a = t \Rightarrow dx = \frac{dt}{\log a}$

$$\begin{aligned} \therefore \int_0^{\infty} \frac{x^a}{a^x} dx &= \int_0^{\infty} e^{-t} \left(\frac{t}{\log a}\right)^a \frac{dt}{\log a} \\ &= \frac{1}{(\log a)^{a+1}} \int_0^{\infty} e^{-t} t^{(a+1)-1} dt = \frac{\Gamma(a+1)}{(\log a)^{a+1}} \end{aligned}$$

iv. We have $\Gamma n = \int_0^{\infty} e^{-t} t^{n-1} dt$

Putting $t = \log \frac{1}{x} \Rightarrow -t = \log x \Rightarrow e^{-t} = x$

Also $dt = -\frac{1}{x} dx$

as $t = 0 \Rightarrow x = 1$, $t = \infty \Rightarrow x = 0$

$$\begin{aligned} \therefore \Gamma n &= \int_1^0 x \left(\log \frac{1}{x}\right)^{n-1} \left(-\frac{1}{x}\right) dx = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx \\ &\Rightarrow \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \Gamma n \end{aligned}$$

II Beta Function

Beta function is defined as:

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx, \quad m, n > 0$$

Important Results

4. Beta function is symmetric i.e. $\beta(m, n) = \beta(n, m)$

$$\begin{aligned}\text{Proof: } \beta(m, n) &= \int_0^1 x^{m-1} (1-x)^{n-1} dx, \quad m, n > 0 \\ &= \int_0^1 (1-x)^{m-1} (1-(1-x))^{n-1} dx, \quad m, n > 0 \\ &\quad \because \int_0^a f(x) dx = \int_0^a f(a-x) dx \\ &= \int_0^1 x^{n-1} (1-x)^{m-1} dx, \quad n, m > 0 \\ &= \beta(n, m)\end{aligned}$$

5. Another definition of Beta function:

$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx, \quad m, n > 0$$

$$\text{Proof: } \beta(m, n) = \int_0^1 y^{m-1} (1-y)^{n-1} dy, \quad m, n > 0$$

$$\text{Putting } y = \frac{1}{1+x}, \quad dy = -\frac{1}{(1+x)^2} dx$$

$$\begin{aligned}\Rightarrow \beta(m, n) &= -\int_\infty^0 \left(\frac{1}{1+x}\right)^{m-1} \left(1 - \frac{1}{1+x}\right)^{n-1} \frac{1}{(1+x)^2} dx, \quad m, n > 0 \\ &= \int_0^\infty \left(\frac{1}{1+x}\right)^{m-1} \left(\frac{x}{1+x}\right)^{n-1} \left(\frac{1}{1+x}\right)^2 dx \\ &= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx \\ &= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \quad \because \beta(m, n) = \beta(n, m)\end{aligned}$$

6. Another form of Beta function is given by:

$$\beta(m, n) = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

Proof: we have $\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

Let $x = \sin^2 \theta \Rightarrow dx = 2 \sin \theta \cos \theta d\theta$

$$\begin{aligned} \therefore \beta(m, n) &= \int_0^{\frac{\pi}{2}} (\sin^2 \theta)^{m-1} (\cos^2 \theta)^{n-1} 2 \sin \theta \cos \theta d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta \end{aligned}$$

7. Relation between Beta Gamma functions:

$$\beta(m, n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}, \quad m, n > 0$$

Proof: Using result 3, $\int_0^\infty e^{-kx} x^{m-1} dx = \frac{\Gamma m}{k^m} \dots \textcircled{1}$

Replacing k by y , we get $\frac{\Gamma m}{y^m} = \int_0^\infty e^{-yx} x^{m-1} dx$

$$\Rightarrow \Gamma m = \int_0^\infty e^{-yx} y^m x^{m-1} dx$$

$$\Rightarrow e^{-y} y^{n-1} \Gamma m = \int_0^\infty e^{-y(1+x)} y^{m+n-1} x^{m-1} dx$$

Integrating both sides with respect to y within limits 0 to ∞

$$\Gamma m \int_0^\infty e^{-y} y^{n-1} dy = \int_0^\infty \int_0^\infty e^{-y(1+x)} y^{m+n-1} x^{m-1} dx dy$$

$$\Rightarrow \Gamma m \Gamma n = \int_0^\infty \left[\int_0^\infty e^{-(1+x)y} y^{(m+n-1)} dy \right] x^{m-1} dx$$

$$\Rightarrow \Gamma m \Gamma n = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}} x^{m-1} dx, \text{ comparing with } \textcircled{1}$$

$$\Rightarrow \frac{\Gamma m \Gamma n}{\Gamma(m+n)} = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$\Rightarrow \frac{\Gamma m \Gamma n}{\Gamma(m+n)} = \beta(m, n), \text{ using result 5}$$

$$\mathbf{8.} \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

Proof: we have $\beta(m, n) = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$

$$\Rightarrow \frac{\Gamma m \Gamma n}{\Gamma(m+n)} = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta \because \beta(m, n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

Replacing $2m - 1$ by p and $2n - 1$ by q

$$\text{i.e } m = \frac{p+1}{2} \text{ and } n = \frac{q+1}{2}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} \dots \textcircled{1}$$

Putting $q = 0$ in $\textcircled{1}$, we get $\int_0^{\frac{\pi}{2}} \sin^p \theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)}$

Putting $p = 0$ in $\textcircled{1}$, we get $\int_0^{\frac{\pi}{2}} \sin^p \theta d\theta = \frac{\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{q+2}{2}\right)}$

9. Duplication formula is given by:

$$\Gamma m \Gamma \left(m + \frac{1}{2} \right) = \frac{\sqrt{\pi} \cdot \Gamma(2m)}{2^{2m-1}}, m > 0$$

Proof: We have $\beta(m, n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$

$$\therefore 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} \quad \dots \textcircled{1}$$

Putting $n = \frac{1}{2}$ on both sides, we get

$$2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta d\theta = \frac{\Gamma m \sqrt{\pi}}{\Gamma(m+\frac{1}{2})} \quad \dots \textcircled{2}$$

Again Putting $n = m$ in $\textcircled{1}$, we get

$$\begin{aligned} \frac{(\Gamma m)^2}{\Gamma(2m)} &= 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2m-1} \theta d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} \left(\frac{2 \sin \theta \cos \theta}{2} \right)^{2m-1} d\theta \\ &= \frac{1}{2^{2m-2}} \int_0^{\frac{\pi}{2}} \sin^{2m-1} 2\theta d\theta \\ &= \frac{1}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1} t dt, \text{ Putting } 2\theta = t \\ &= \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} \sin^{2m-1} t dt \end{aligned}$$

$$\therefore \int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx, \text{ if } f(2a-x) = f(x)$$

$$\Rightarrow \frac{(\Gamma m)^2}{\Gamma(2m)} = \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta d\theta$$

$$\Rightarrow 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta d\theta = \frac{2^{2m-1}(\Gamma m)^2}{\Gamma(2m)} \quad \dots \textcircled{3}$$

Comparing $\textcircled{2}$ and $\textcircled{3}$, we get $\frac{\Gamma m \sqrt{\pi}}{\Gamma(m+\frac{1}{2})} = \frac{2^{2m-1}(\Gamma m)^2}{\Gamma(2m)}$

$$\Rightarrow \Gamma m \Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi} \Gamma(2m)}{2^{2m-1}}$$

$$\mathbf{10. \Gamma n \Gamma(1 - n) = \frac{\pi}{\sin n\pi}, \mathbf{0 < n < 1}}$$

Proof: we have $\beta(m, n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx, m, n > 0$

$$\Rightarrow \frac{\Gamma m \Gamma n}{\Gamma(m+n)} = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Putting $m = 1 - n$ on both sides, we get

$$\Gamma n \Gamma(1 - n) = \int_0^{\infty} \frac{x^{n-1}}{1+x} dx$$

Putting $x = e^t, dx = e^t dt$

As $x \rightarrow 0, t \rightarrow -\infty$ and as $x \rightarrow \infty, t \rightarrow \infty$

$$\therefore \Gamma n \Gamma(1 - n) = \int_{-\infty}^{\infty} \frac{e^{nt}}{1+e^t} dt$$

Now by using complex integration, we have:

$$\int_{-\infty}^{\infty} \frac{e^{nt}}{1+e^t} dt = \frac{\pi}{\sin n\pi}, \quad 0 < n < 1$$

$$\therefore \Gamma n \Gamma(1 - n) = \frac{\pi}{\sin n\pi}, \quad 0 < n < 1$$

Example 3 Evaluate *i.* $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^{\frac{5}{2}} x dx$ *ii.* $\int_0^{\frac{\pi}{2}} \sin^{10} x dx$ *iii.* $\int_0^{\frac{\pi}{2}} \sqrt{\tan \theta} + \sqrt{\sec \theta} d\theta$

$$\textit{iv.} \int_0^2 x^{m-1} (2-x)^{n-1} dx \quad \textit{v.} \int_0^{\pi} \sin^2 \theta (1 + \cos \theta)^4 d\theta \quad \textit{vi.} \int_0^1 x^{m-1} \left(\log \frac{1}{x}\right)^{n-1} dx$$

$$\textbf{Solution:} \quad \textit{i.} \int_0^{\frac{\pi}{2}} \sin^3 x \cos^{\frac{5}{2}} x dx = \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{\frac{5}{2}+1}{2}\right)}{2\Gamma\left(\frac{3+\frac{5}{2}+2}{2}\right)} = \frac{\Gamma 2 \Gamma\left(\frac{7}{4}\right)}{2\Gamma\left(\frac{15}{4}\right)}$$

$$= \frac{1 \cdot \Gamma\left(\frac{7}{4}\right)}{2 \cdot \frac{11}{4} \cdot \frac{7}{4} \cdot \Gamma\left(\frac{7}{4}\right)} = \frac{8}{77} \quad \because \Gamma 2 = 1! = 1, \text{ also } \Gamma(n+1) = n\Gamma n$$

$$\textit{ii.} \int_0^{\frac{\pi}{2}} \sin^{10} x dx = \frac{\Gamma\left(\frac{11}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma 6} = \frac{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{240} = \frac{945\pi}{7680}$$

$$\because \Gamma 6 = 5! = 120, \text{ also } \Gamma(n+1) = n\Gamma n$$

$$= \frac{63\pi}{512} \quad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\textit{iii.} \int_0^{\frac{\pi}{2}} \sqrt{\tan \theta} + \sqrt{\sec \theta} d\theta = \int_0^{\frac{\pi}{2}} (\sin^{\frac{1}{2}} \theta \cos^{-\frac{1}{2}} \theta + \cos^{-\frac{1}{2}} \theta) d\theta$$

$$= \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} + \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+2}{2}\right)}$$

$$\begin{aligned}
&= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma 1} + \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{3}{4}\right)} \\
&= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left\{\Gamma\left(\frac{3}{4}\right) + \frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}\right\}
\end{aligned}$$

iv. Let $I = \int_0^2 x^{m-1} (2-x)^{n-1} dx$

Putting $x = 2\sin^2 \theta$,

$$I = \int_0^{\frac{\pi}{2}} 2^{m-1} \sin^{2m-2} \theta \cdot 2^{n-1} \cos^{2n-2} \theta \cdot 2 \sin \theta \cos \theta d\theta$$

$$\Rightarrow I = 2^{m+n-2} \cdot 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta = 2^{m+n-2} \beta(m, n)$$

$$\therefore 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta = \beta(m, n)$$

v. Let $I = \int_0^{\pi} \sin^2 \theta (1 + \cos \theta)^4 d\theta$

$$= \int_0^{\pi} \left(2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right)^2 \left(2 \cos^2 \frac{\theta}{2}\right)^4 d\theta$$

$$= 64 \int_0^{\pi} \sin^2 \frac{\theta}{2} \cos^{10} \frac{\theta}{2} d\theta$$

Putting $\frac{\theta}{2} = x$, $d\theta = 2x dx$

$$= 128 \int_0^{\frac{\pi}{2}} \sin^2 x \cos^{10} x dx$$

$$= \frac{64 \cdot \Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{11}{2}\right)}{\Gamma 7} = \frac{64 \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{720} = \frac{21\pi}{16}$$

$$\because \Gamma 7 = 6! = 720, \text{ also } \Gamma(n+1) = n\Gamma n$$

$$vi. \text{ Let } I = \int_0^1 x^{m-1} \left(\log \frac{1}{x} \right)^{n-1} dx$$

$$\text{Putting } \log \frac{1}{x} = t \text{ or } x = e^{-t} \Rightarrow dx = -e^{-t} dt,$$

$$\begin{aligned} I &= - \int_{\infty}^0 e^{-(m-1)t} t^{n-1} e^{-t} dt \\ &= \int_0^{\infty} e^{-mt} t^{n-1} dt \end{aligned}$$

$$\text{Putting } mt = y$$

$$I = \frac{1}{m} \int_0^{\infty} e^{-y} \left(\frac{y}{m} \right)^{n-1} dy = \frac{1}{m^n} \int_0^{\infty} e^{-y} y^{n-1} dy = \frac{\Gamma n}{m^n}$$

Example 4 Prove that i. $\beta(m, n) = \beta(m+1, n) + \beta(m, n+1)$

$$ii. \frac{\beta(m+1, n)}{\beta(m, n)} = \frac{m}{m+n}$$

$$iii. \beta\left(m, \frac{1}{2}\right) = 2^{2m-1} \beta(m, m)$$

Solution: i. R.H.S. = $\beta(m+1, n) + \beta(m, n+1)$

$$\begin{aligned} &= \frac{\Gamma(m+1)\Gamma n}{\Gamma(m+n+1)} + \frac{\Gamma m \Gamma(n+1)}{\Gamma(m+n+1)} \\ &= \frac{m\Gamma m \cdot \Gamma n + \Gamma m \cdot n\Gamma n}{\Gamma(m+n+1)} \\ &= \frac{\Gamma m \cdot \Gamma n (m+n)}{(m+n)\Gamma(m+n)} \\ &= \frac{\Gamma m \Gamma n}{\Gamma(m+n)} = \beta(m, n) = \text{L.H.S.} \end{aligned}$$

$$\begin{aligned}
 \text{ii. L.H.S.} &= \frac{\beta(m+1, n)}{\beta(m, n)} = \frac{\Gamma(m+1)\Gamma n}{\Gamma(m+n+1)} \cdot \frac{\Gamma(m+n)}{\Gamma m \Gamma n} \\
 &= \frac{m\Gamma m \Gamma n}{(m+n)\Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma m \Gamma n} = \frac{m}{m+n} = \text{R.H.S.}
 \end{aligned}$$

$$\text{iii. We have } \beta\left(m, \frac{1}{2}\right) = \frac{\Gamma m \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m + \frac{1}{2}\right)} \quad \dots \textcircled{1}$$

$$\text{Again, by Duplication formula } \Gamma m \Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi} \Gamma(2m)}{2^{2m-1}}$$

$$\therefore \Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi} \Gamma(2m)}{2^{2m-1} \Gamma m} \quad \dots \textcircled{2}$$

$$\begin{aligned}
 \text{Using } \textcircled{2} \text{ in } \textcircled{1}, \text{ we get } \beta\left(m, \frac{1}{2}\right) &= \frac{\Gamma m \Gamma\left(\frac{1}{2}\right) 2^{2m-1} \Gamma m}{\sqrt{\pi} \Gamma(2m)} \\
 &= 2^{2m-1} \frac{\Gamma m \Gamma m}{\Gamma(2m)} \quad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \\
 &= 2^{2m-1} \beta(m, m)
 \end{aligned}$$

Example 5 Express the following integrals in terms of Beta function

$$\text{i. } \int_0^1 x^m (1-x^2)^n dx \quad \text{ii. } \int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx$$

$$\text{Solution: i. Let } I = \int_0^1 x^m (1-x^2)^n dx = \frac{1}{2} \int_0^1 x^{m-1} (1-x^2)^n 2x dx$$

$$\text{Putting } x^2 = y \Rightarrow 2x dx = dy$$

$$\Rightarrow I = \frac{1}{2} \int_0^1 y^{\frac{m-1}{2}} (1-y)^n dy$$

$$= \frac{1}{2} \int_0^1 y^{\frac{m+1}{2}-1} (1-y)^{(n+1)-1} dy$$

$$= \frac{1}{2} \beta\left(\frac{m+1}{2}, n+1\right)$$

ii. Let $I = \int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx = \frac{1}{5} \int_0^1 x^{-2} (1-x^5)^{-\frac{1}{2}} 5x^4 dx$

Putting $x^5 = y \Rightarrow 5x^4 dx = dy$

$$\Rightarrow I = \frac{1}{5} \int_0^1 y^{-\frac{2}{5}} (1-y)^{-\frac{1}{2}} dy$$

$$= \frac{1}{5} \int_0^1 y^{\frac{3}{5}-1} (1-y)^{\frac{1}{2}-1} dy = \frac{1}{5} \beta\left(\frac{3}{5}, \frac{1}{2}\right)$$

Example 6 Prove that i. $\Gamma\left(\frac{3}{2}-x\right) \Gamma\left(\frac{3}{2}+x\right) = \left(\frac{1}{4}-x^2\right) \pi \sec \pi x$

ii. $\int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1} \beta(m+1, n+1)$

Solution i. L.H.S. = $\Gamma\left(\frac{3}{2}-x\right) \Gamma\left(\frac{3}{2}+x\right)$

$$= \left(\frac{1}{2}-x\right) \Gamma\left(\frac{1}{2}-x\right) \cdot \left(\frac{1}{2}+x\right) \Gamma\left(\frac{1}{2}+x\right)$$

$$\because \Gamma(n+1) = n\Gamma n$$

$$= \left(\frac{1}{4}-x^2\right) \Gamma\left(\frac{1}{2}+x\right) \Gamma\left(1-\left(\frac{1}{2}+x\right)\right)$$

$$\because \Gamma\left(\frac{1}{2}-x\right) = \Gamma\left(1-\left(\frac{1}{2}+x\right)\right)$$

$$= \left(\frac{1}{4}-x^2\right) \frac{\pi}{\sin\left(\frac{1}{2}+x\right)\pi}, 0 < \frac{1}{2}+x < 1$$

$$\begin{aligned}
& \because \Gamma n \Gamma(1-n) = \frac{\pi}{\sin n\pi}, 0 < n < 1 \\
& = \left(\frac{1}{4} - x^2\right) \frac{\pi}{\cos \pi x} \\
& = \left(\frac{1}{4} - x^2\right) \pi \cdot \sec \pi x, -\frac{1}{2} < x < \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\text{ii. Let } I &= \int_a^b (x-a)^m (b-x)^n dx \\
&= \int_0^{b-a} y^m (b-a-y)^n dy \quad \text{By putting } x-a=y \\
&= \int_0^1 (b-a)^m t^m (b-a-(b-a)t)^n (b-a) dt
\end{aligned}$$

By putting $y = (b-a)t$

$$\begin{aligned}
I &= \int_0^1 x^m (1-x^n)^p dx = \int_0^1 (b-a)^m t^m (b-a)^n (1-t)^n (b-a) dt \\
&= (b-a)^{m+n+1} \int_0^1 t^m (1-t)^n dt \\
&= (b-a)^{m+n+1} \int_0^1 t^{(m+1)-1} (1-t)^{(n+1)-1} dt \\
&= (b-a)^{m+n+1} \beta(m+1, n+1)
\end{aligned}$$

Example 7 Express the integral $\int_0^1 x^m (1-x^n)^p dx$ in terms of gamma function and hence evaluate

$$\int_0^1 x^{\frac{3}{2}} \left(1-x^{\frac{1}{2}}\right)^{\frac{1}{2}} dx$$

Solution: Let $I = \int_0^1 x^m (1-x^n)^p dx$

Putting $x^n = t$, so that $nx^{n-1}dx = dt$, we get

$$I = \frac{1}{n} \int_0^1 t^{\frac{m}{n}} (1-t)^p t^{-\frac{n-1}{n}} dt = \frac{1}{n} \int_0^1 t^{\frac{m+1}{n}-1} (1-t)^{p+1-1} dt$$

$$\therefore I = \int_0^1 x^m (1-x^n)^p dx = \frac{1}{n} \beta\left(\frac{m+1}{n}, p+1\right) = \frac{1}{n} \frac{\Gamma\left(\frac{m+1}{n}\right) \Gamma(p+1)}{\Gamma\left(\frac{m+1}{n} + p+1\right)} \dots \textcircled{1}$$

Putting $m = \frac{3}{2}$, $n = \frac{1}{2}$, $p = \frac{1}{2}$ in $\textcircled{1}$, we get

$$\begin{aligned} \int_0^1 x^{\frac{3}{2}} \left(1 - x^{\frac{1}{2}}\right)^{\frac{1}{2}} dx &= 2\beta\left(\frac{\frac{3}{2}+1}{\frac{1}{2}}, \frac{1}{2} + 1\right) = 2\beta\left(5, \frac{3}{2}\right) \\ &= \frac{2\Gamma(5)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(5+\frac{3}{2}\right)} = \frac{2.4! \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{13}{2}\right)} = \frac{48\Gamma\left(\frac{3}{2}\right)}{\frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \Gamma\left(\frac{3}{2}\right)} = \frac{512}{3465} \end{aligned}$$

Example 8 Evaluate i. $\int_0^\infty x^{n-1} \cos ax \, dx$ ii. $\int_0^\infty x^{n-1} \sin ax \, dx$

Solution: let $I = \int_0^\infty x^{n-1} e^{-iax} dx = \int_0^\infty x^{n-1} (\cos ax - i \sin ax) \, dx$

Putting $iax = t$, $dx = \frac{dt}{ia}$

$$\begin{aligned} \therefore I &= \frac{1}{ia} \int_0^\infty e^{-t} \left(\frac{t}{ia}\right)^{n-1} dt = \frac{1}{i^n a^n} \int_0^\infty e^{-t} t^{n-1} dt = \frac{\Gamma n}{i^n a^n} \\ &= \frac{\Gamma n}{a^n} (-i)^n = \frac{\Gamma n}{a^n} \left(\cos \frac{\pi}{2} - i \sin \frac{\pi}{2}\right)^n \\ &= \frac{\Gamma n}{a^n} \left(\cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2}\right) \end{aligned}$$

$$\therefore \int_0^\infty x^{n-1} (\cos ax - i \sin ax) \, dx = \frac{\Gamma n}{a^n} \left(\cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2}\right)$$

Comparing real and imaginary parts we get,

$$i. \int_0^{\infty} x^{n-1} \cos ax \, dx = \frac{\Gamma n}{a^n} \cos \frac{n\pi}{2}$$

$$ii. \int_0^{\infty} x^{n-1} \sin ax \, dx = \frac{\Gamma n}{a^n} \sin \frac{n\pi}{2}$$

Exercise

$$1. \text{ Show that } \int_0^{\frac{\pi}{2}} \tan^n x \, dx = \frac{\pi}{2} \sec \left(\frac{n\pi}{2} \right), \quad 0 < n < 1$$

$$2. \text{ Given that } \Gamma \left(\frac{8}{5} \right) = 0.8935, \text{ find the values of } \Gamma \left(-\frac{5}{2} \right) \text{ and } \Gamma \left(-\frac{12}{5} \right)$$

$$3. \text{ Find } i. \beta \left(\frac{3}{2}, \frac{1}{2} \right) \quad ii. \beta \left(\frac{4}{3}, \frac{5}{3} \right)$$

$$4. \text{ Evaluate } \int_0^1 x^2 (1 - x^2)^4 \, dx$$

$$5. \text{ Prove that } \Gamma n = 2 \int_0^{\infty} e^{-x^2} x^{2n-1} \, dx$$

$$6. \text{ Prove that for } a, b > 0$$

$$i. \int_0^{\infty} x^{n-1} e^{-ax} \cos bx \, dx = \frac{\Gamma n}{(a^2 + b^2)^{n/2}} \cos \left(n \tan^{-1} \frac{b}{a} \right)$$

$$ii. \int_0^{\infty} x^{n-1} e^{-ax} \sin bx \, dx = \frac{\Gamma n}{(a^2 + b^2)^{n/2}} \sin \left(n \tan^{-1} \frac{b}{a} \right)$$

$$7. \text{ Show that } \frac{\beta(m+1, n)}{m} = \frac{\beta(m, n)}{m+n+1}$$

$$8. \text{ Show that } \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin x}} \, dx \cdot \int_0^{\frac{\pi}{2}} \sqrt{\sin x} \, dx = \pi$$

$$9. \text{ Show that } \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} \, dx = \beta(m, n)$$

10. Prove that $\beta\left(m, \frac{1}{2}\right) = 2^{2m-1} \cdot \beta(m, m)$

Answers

2. $-\frac{8}{15}\sqrt{\pi}, -1.108$

3. $i \cdot \frac{\pi}{2} \quad ii \cdot \frac{2\pi}{9\sqrt{3}}$

4. $\frac{128}{3465}$